

Lower Bound Approximation to Basket Option Values for Local Volatility Jump-Diffusion Models

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Abstract. In this paper we derive an easily computed approximation of Rogers and Shi's lower bound for a local volatility jump-diffusion model and then use it to approximate European basket option values. If the local volatility function is time independent then there is a closed-form expression for the approximation. Numerical tests show that the lower bound approximation is fast and accurate in comparison with the Monte Carlo method, the partial-exact approximation method and the asymptotic expansion method.

Keywords. Basket options valuation, local volatility jump-diffusion model, lower bound approximation, second order asymptotic expansion.

1 Introduction

It is in general difficult to value basket options due to lack of analytic characterization of the distribution of the underlying basket asset process. Research is mainly focused on developing fast and accurate approximation methods and finding tight lower and upper bounds for basket option values. In the Black-Scholes setting Curran (1994) and Rogers and Shi (1995) derive a lower bound for Asian options by conditioning random variables and Jensen's inequality. Deelstra et al. (2004) obtain the bounds for basket options with the comonotonicity approach. In affine Lévy models one can derive lower bounds numerically for arithmetic Asian options based on the characteristic function and the method developed in Duffie et al. (2000), see Albrecher et al. (2008). Deelstra et al. (2010) provide a good overview of the recent development in finding and computing the bounds.

It is known that Rogers and Shi's lower bound is generally tight and is one of the most accurate approximations to basket option values. The lower bound can be calculated exactly in the Black-Scholes framework. Xu and Zheng (2009) show that the lower bound can also be calculated exactly in a special jump-diffusion model with constant volatility and two types of Poisson jumps (systematic and idiosyncratic jumps). The usefulness of Rogers and Shi's lower bound depends crucially on one's ability of finding some highly correlated random variables to the basket value and computing the conditional expectation exactly. It is difficult to extend Rogers and Shi's lower bound to more general models such as local volatility models due to lack of explicitly known distributions for models with non-constant volatilities, see Albrecher et al. (2008). To the best of our knowledge, Rogers and Shi's lower bound for models with local volatilities has not been discussed in the literature.

In this paper we aim to find a good approximation to Rogers and Shi's lower bound for a local volatility jump-diffusion model and then use it to approximate the European basket option values. We first apply the second order asymptotic expansion (see Benhamou et al. (2009)) to approximate the basket asset value, then choose a normal variable and a Poisson variable as conditioning variables which are highly correlated to the basket asset value, and finally apply the conditional expectation results of multiple Wiener-Itô integrals (see Kunitomo and Takahashi

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(2001)) to approximate Rogers and Shi's lower bound. The main contribution of the paper is the derivation of an approximation to Rogers and Shi's lower bound for local volatility jump-diffusion models. We suggest an easily implemented algorithm to compute the lower bound approximation. If the local volatility function is time independent then there is a closed-form expression for the approximation. Numerical tests show that the lower bound approximation is fast and accurate in most cases in comparison with the Monte Carlo method, the partial-exact approximation method (Xu and Zheng (2009)) and the asymptotic expansion method (Xu and Zheng (2010)).

The paper is organized as follows. Section 2 formulates the basket local volatility jump-diffusion model and explains some known methods in pricing European basket options. Section 3 applies the second order asymptotic expansion to derive an approximation to Roger and Shi's lower bound which can be easily computed. Section 4 compares the numerical performance of the lower bound approximation with other methods in pricing basket options. Section 5 concludes.

2 Model

Assume $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$ is a filtered risk-neutral probability space and \mathcal{F}_t is the augmented natural filtration generated by correlated Brownian motions W_1, \dots, W_n with correlation matrix (ρ_{ij}) and a Poisson process N with intensity λ . Assume Brownian motions W_1, \dots, W_n and Poisson process N are independent to each other. Assume the portfolio is composed of n assets and the discounted asset prices S_1, \dots, S_n satisfy the stochastic differential equations

$$dS_i(t) = \sigma_i(t, S_i(t))dW_i(t) + h_i S_i(t-)dM(t), \quad (1)$$

where $h_i > -1$ are constant jump sizes and $M(t) = N(t) - \lambda t$ is the compensated Poisson process. The discounted basket value at time T is given by

$$S(T) = \sum_{i=1}^n w_i S_i(T),$$

where w_i are positive constant weights. The European basket call option price at time 0 is given by

$$C(T, K) = \mathbf{E}[(S(T) - K)^+]$$

Almost all work in the literature on Asian or basket options valuation assume the underlying asset prices follow lognormal processes, which corresponds to $\sigma_i(t, S_i) = \sigma_i S_i$ with σ_i being positive constant and $h_i = 0$ for all i in (1). Since Asian options are similar to basket options, we do not differentiate these two types of options, even though some techniques are originally developed for Asian options. We now review some well-known approaches in approximation and error bound estimation for the pure diffusion case.

Levy (1992) approximates the basket value $S(T)$ with a lognormal variable which has the same first two moments as those of $S(T)$ and derives an approximate closed-form pricing formula for $C(T, K)$. The result is very good when maturity T and volatilities σ_i are relatively small. The performance deteriorates as T or σ_i increases. Curran (1994) introduces the idea of conditioning variables. Assume Λ is a random variable which has strong correlation with $S(T)$ and satisfies that $S(T) \geq K$ whenever $\Lambda \geq d_\Lambda$ for some constant d_Λ . The basket option price can be decomposed as

$$\mathbf{E}[(S(T) - K)^+] = \mathbf{E}[(S(T) - K)1_{[\Lambda \geq d_\Lambda]}] + \mathbf{E}[(S(T) - K)^+ 1_{[\Lambda < d_\Lambda]}].$$

Curran (1994) chooses Λ a normal variable (geometric average) and finds the closed-form expression for the the first part and uses the lognormal variable and the conditional moment matching

to find the approximate value of the second part. Rogers and Shi (1995) use the conditioning variable Λ to derive the lower and upper bounds to the basket option price $C(T, K)$ as

$$LB = \mathbf{E} [(\mathbf{E}[S(T)|\Lambda] - K)^+]$$

and

$$UB = LB + \frac{1}{2} \mathbf{E} [\text{var}(S(T)|\Lambda) 1_{[\Lambda < d_\Lambda]}]^{\frac{1}{2}} \mathbf{E}[1_{[\Lambda < d_\Lambda]}]^{\frac{1}{2}}.$$

These bounds may be computed analytically for specific conditioning variables Λ . Numerical tests show that the lower bound is normally very tight to the true (Monte Carlo) value.

Xu and Zheng (2009) extend the partial exact approximation method of Curran (1994) to jump-diffusion processes (1) with local volatility functions $\sigma_i(t, S_i) = \sigma_i S_i$ for all i . (The jump part of the model in Xu and Zheng (2009) is more general than (1) with both common and individual jumps.) Asset prices $S_i(T)$ at time T have closed form expressions

$$S_i(T) = S_i(0) \exp \left(-\frac{1}{2} \sigma_i^2 T - h_i \lambda T + \sigma_i W_i(T) + N(T) \ln(1 + h_i) \right), \quad (2)$$

where $S_i(0)$ are asset prices at time 0, for $i = 1, \dots, n$. The conditioning variable Λ is chosen to be

$$\Lambda = mN(T) + \sigma W(T),$$

where $N(T)$ is a Poisson variable with parameter λT , $W(T)$ a standard normal variable, $N(T)$ and $W(T)$ are independent to each other, $m = \sum_{i=1}^n a_i \ln(1 + h_i)$, $\sigma^2 = \sum_{i,j=1}^n a_i a_j \rho_{ij} \sigma_i \sigma_j T$, and $a_i = w_i S_i(0) \exp((-\frac{1}{2} \sigma_i^2 - h_i \lambda)T)$ for $i = 1, \dots, n$. We then have $S(T) \geq K$ whenever $\Lambda \geq K - c$, where $c = \sum_{i=1}^n a_i$. For $N(T) = k$ and $W(T) = y$ it is easy to find

$$E[S(T)|\Lambda = mk + \sigma y] = \sum_{i=1}^n a_i \exp \left(\frac{1}{2} (\sigma_i^2 T - R_i^2) + R_i y + k \ln(1 + h_i) \right),$$

where $R_i = \frac{1}{\sigma} \sum_{j=1}^n a_j \rho_{ij} \sigma_i \sigma_j T$. The lower bound can be computed from the expression

$$LB = \sum_{k=0}^{\infty} p_k \int_{-\infty}^{\infty} (E[S(T)|\Lambda = mk + \sigma y] - K)^+ d\Phi(y),$$

where $p_k = P(N(T) = k)$ and Φ is the cumulative distribution function of a standard normal variable. Xu and Zheng (2009) suggest a partial exact approximation (PEA) to the basket option value $C(T, K)$ by the following value

$$C^A(T, K) := \mathbf{E}[(S(T) - K)^+ 1_{[\Lambda \geq K - c]}] + \sum_{i=1}^3 q_i \mathbf{E}[(\mathbf{E}[S(T)|\Lambda] + \alpha_i - K)^+ 1_{[\Lambda < K - c]}],$$

where $q_1 = 1/6$, $q_2 = 2/3$, $q_3 = 1/6$, and $\alpha_1 = -\sqrt{3}\epsilon_0$, $\alpha_2 = 0$, $\alpha_3 = \sqrt{3}\epsilon_0$, ϵ_0 is some constant depending on the conditioning variable Λ and the conditional second moment of $S(T)$ given Λ . The lower bound plays a dominant role in the approximation with a weight 2/3, the other two parts with a weight 1/6 each are the second moment adjustment to the lower bound. Xu and Zheng (2009) show that $LB \leq C^A(T, K) \leq UB$, which provides the error bounds for the approximate basket option value. The PEA method is fast and accurate in comparison with some other well known numerical schemes for basket options, see Xu and Zheng (2009) for details. A key condition for the PEA method is that one must know the closed form expressions of asset prices $S_i(T)$ as in (2) for $i = 1, \dots, n$. This is impossible for general local volatility functions.

Xu and Zheng (2010) discuss the basket options pricing for asset model (1) with the same jump sizes $h_i = h$ for $i = 1, \dots, n$. Since the PEA method cannot be applied for general local volatility

functions $\sigma_i(t, S_i)$, Xu and Zheng (2010) first reduce the dimensionality of the problem by directly working on the portfolio asset value $S(t)$ which satisfies the following stochastic volatility jump diffusion model

$$dS(t) = V(t)dW(t) + hS(t-)dM(t)$$

with the initial price $S(0) = \sum_{i=1}^n w_i S_i(0)$, where W is a standard Brownian motion, independent of M , and

$$V(t)^2 := \sum_{i,j=1}^n w_i w_j \sigma_i(t, S_i(t)) \sigma_j(t, S_j(t)) \rho_{ij}.$$

The European basket call option price $C(T, K)$ at time 0 satisfies a partial integral differential equation (PIDE)

$$C_T(T, K) = \lambda h K C_K(T, K) + \frac{1}{2} \sigma(T, K)^2 C_{KK}(T, K) + \lambda(h+1) \left(C(T, \frac{K}{h+1}) - C(T, K) \right)$$

with the initial condition $C(0, K) = (S(0) - K)^+$ and the local variance function

$$\sigma(T, K)^2 = \mathbf{E}[V(T)^2 | S(T) = K].$$

The main difficulty is to compute the conditional expectation as asset prices $S_i(T)$ have no closed-form expressions. Following the first order asymptotic expansion methods of Kunitomo and Takahashi (2001) and Benhamou et al. (2009), Xu and Zheng (2010) show that the unknown local variance function $\sigma(T, K)^2$ can be approximated by

$$\sigma(T, K)^2 \approx a(T) + b(T)(K - S(0)), \quad (3)$$

where $a(T) = \sum_{i,j=1}^n w_i w_j \rho_{ij} p_i p_j$, $b(T) = \sum_{i,j=1}^n \frac{1}{\sigma_c^2} w_i w_j \rho_{ij} p_i p_j \left(\frac{q_i}{p_i} C_i + \frac{q_j}{p_j} C_j \right)$, $p_i = \sigma_i(T, S_i(0))$, $q_i = \frac{\partial}{\partial S_i(0)} \sigma_i(T, S_i(0))$, $C_i = \sum_{j=1}^n w_j \left[\rho_{ij} \left(\int_0^T \sigma_i(t, S_i(0)) \sigma_j(t, S_j(0)) dt \right) \right]$, and $\sigma_c^2 = \sum_{i=1}^n w_i C_j$. One can find the basket option price by solving the PIDE with the implicit-explicit finite difference method. The method is less accurate than the PEA method when the local volatility function is the Black-Scholes type, but is more general and can handle general local volatility functions. The main limitations of the method are that the approximate variance function in (3) is not always positive, which can cause numerical errors in solving the PIDE, and that it requires the same jump size for all assets, which is unrealistic for a general asset portfolio.

3 Lower Bound Approximation

The asymptotic expansion for basket options pricing in a general diffusion model is discussed in Kunitomo and Takahashi (2001) in which the valuation of conditional expectations is a necessary step to obtain the characteristic function of the basket value. In this paper we use the same method to expand the parameterized asset price processes to the second order, see Benhamou et al. (2009), and apply the conditional expectation results of multiple Wiener-Itô integrals directly to approximate the Rogers and Shi's lower bound for the jump-diffusion model (1). For $\epsilon \in [0, 1]$ define

$$dS_i^\epsilon(t) = \epsilon \sigma_i(t, S_i^\epsilon(t)) dW_i(t) + \epsilon h_i S_i^\epsilon(t-) dM(t) \quad (4)$$

with initial condition $S_i^\epsilon(0) = S_i(0)$. Note that $S_i^1(T) = S_i(T)$. Define

$$S_i^{(k)}(t) := \frac{\partial^k S_i^\epsilon(t)}{\partial \epsilon^k} \Big|_{\epsilon=0} \quad \text{and} \quad \sigma_i^{(k)}(t) := \frac{\partial^k \sigma_i(t, S_i^\epsilon(t))}{\partial (S_i^\epsilon)^k} \Big|_{\epsilon=0}$$

for $k = 0, 1, \dots$. It is obvious that $S_i^{(0)}(t) = S_i^0(t) = S_i(0)$ and $\sigma_i^{(0)}(t) = \sigma_i(t, S_i(0))$ for all $t \geq 0$ and $i = 1, \dots, n$. In particular, if local volatility functions σ_i are time independent then functions $\sigma_i^{(k)}$ are all constants for $i = 1, \dots, n$, which may simplify considerably some computations. An important case is the constant elasticity of variance (CEV) model, where $\sigma_i(t, S_i) = \alpha_i S_i^{\beta_i}$ for $i = 1, \dots, n$. To simplify the notations in subsequent discussions we denote by

$$\tilde{\sigma}_i^{(k)}(t) = \sum_{j=1}^n w_j \sigma_i^{(k)}(t) \sigma_j^{(0)}(t) \rho_{ij}, \quad k = 0, 1.$$

The second order asymptotic expansion around $\epsilon = 0$ for $S_i^\epsilon(t)$ is

$$S_i^\epsilon(T) \approx S_i^{(0)}(T) + S_i^{(1)}(T)\epsilon + \frac{1}{2}S_i^{(2)}(T)\epsilon^2.$$

Expand (4) to the second order, we have

$$\begin{aligned} dS_i^{(1)}(t) &= \sigma_i^{(0)}(t)dW_i(t) + h_i S_i(0)dM(t), \\ dS_i^{(2)}(t) &= 2\sigma_i^{(1)}(t)S_i^{(1)}(t)dW_i(t) + 2h_i S_i^{(1)}(t-)dM(t), \end{aligned}$$

with the initial conditions $S_i^{(1)}(0) = S_i^{(2)}(0) = 0$. Therefore,

$$S_i^{(1)}(t) = -\lambda h_i S_i(0)t + \int_0^t \sigma_i^{(0)}(s)dW_i(s) + h_i S_i(0)N(t) \quad (5)$$

$$S_i^{(2)}(t) = -2\lambda h_i \int_0^t S_i^{(1)}(s)ds + 2 \int_0^t \sigma_i^{(1)}(s)S_i^{(1)}(s)dW_i(s) + 2h_i \int_0^t S_i^{(1)}(s-)dN(s) \quad (6)$$

for $0 \leq t \leq T$. Letting $\epsilon = 1$ we may approximate the basket value $S(T)$ by

$$S(T) \approx S^A(T) := S(0) + S^{(1)}(T) + \frac{S^{(2)}(T)}{2}, \quad (7)$$

where $S^{(j)}(T) := \sum_{i=1}^n w_i S_i^{(j)}(T)$ for $j = 1, 2$.

Since there are no closed-form expressions for $S_i(T)$ and $S(T)$, it is difficult to compute the lower bound. If we approximate $S(T)$ by $S^A(T)$, defined in (7), we may be able to compute the conditional expectation $\mathbf{E}[S^A(T)|\Lambda]$ for some conditioning variable Λ and then to approximate the lower bound. We therefore propose that

$$LB \approx LBA := \mathbf{E}[(\mathbf{E}[S^A(T)|\Lambda] - K)^+]. \quad (8)$$

The next step is to choose the conditioning variable for the approximation. In the Black-Scholes framework, it is without exception to choose normal random variable as the conditioning variable, as the asset price is log-normally distributed. It is not clear what one should choose for general jump-diffusion models as there are no closed form solutions for underlying asset prices. With the insight from Xu and Zheng (2009) we choose $\Lambda = (N(T), \Delta(T))$ for the model, where $\Delta(T) = \sum_{j=1}^n w_j \int_0^T \sigma_j^{(0)}(t)dW_j(t)$ is a normal variable with mean 0 and variance $v^2 = \sum_{i=1}^n w_i \int_0^T \tilde{\sigma}_i^{(0)}(t)dt$, and $N(T)$ is a Poisson variable with parameter λT . From (8) and (7) we have

$$LBA = \sum_{k=0}^{\infty} p_k \int_{-\infty}^{\infty} \left[\left(S(0) + \mathbf{E}[S^{(1)}(T)|\Lambda = (k, vx)] + \mathbf{E}\left[\frac{S^{(2)}(T)}{2}|\Lambda = (k, vx)\right] - K \right)^+ \right] d\Phi(x). \quad (9)$$

Since $S^{(j)}(T) = \sum_{i=1}^n w_i S_i^{(j)}(T)$, we only need to find $\mathbf{E}[S_i^{(j)}(T)|\Lambda = (k, vx)]$ for $j = 1, 2$ and $i = 1, \dots, n$. We first calculate $\mathbf{E}[S_i^{(1)}(T)|\Lambda = (k, vx)]$.

$$\begin{aligned} & \mathbf{E}[S_i^{(1)}(T)|\Lambda = (k, vx)] \\ &= -\lambda h_i S_i(0)T + \mathbf{E}[h_i S_i(0)N(T)|N(T) = k] + \mathbf{E}\left[\int_0^T \sigma_i^{(0)}(t)dW_i(t)|\Delta(T) = vx\right] \\ &= -\lambda h_i S_i(0)T + h_i S_i(0)k + \frac{1}{v} \left(\int_0^T \tilde{\sigma}_i^{(0)}(t)dt\right) x. \end{aligned}$$

The valuation of $\mathbf{E}[\frac{S^{(2)}(T)}{2}|\Lambda = (k, vx)]$ is more involved. Since $S_i^{(2)}(T)$ is the sum of three terms in (6) we may find the conditional expectation of each term by substituting $S_i^{(1)}(t)$ in (5) into the integrands and then computing three conditional expectations. We now derive them one by one. The first term can be written as

$$(-\lambda h_i) \mathbf{E}\left[\int_0^T S_i^{(1)}(t)dt|\Lambda = (k, vx)\right] = A_1 + A_2 + A_3,$$

where

$$\begin{aligned} A_1 &= (-\lambda h_i) \mathbf{E}\left[\int_0^T (-\lambda h_i S_i(0)t)dt|\Lambda = (k, vx)\right] \\ &= \frac{1}{2} S_i(0) \lambda^2 h_i^2 T^2 \\ A_2 &= (-\lambda h_i) \mathbf{E}\left[\int_0^T \left(\int_0^t \sigma_i^{(0)}(s)dW_i(s)\right)dt|\Lambda = (k, vx)\right] \\ &= (-\lambda h_i) \mathbf{E}\left[\int_0^T (T-t) \sigma_i^{(0)}(t)dW_i(t)|\Delta(T) = vx\right] \\ &= (-\lambda h_i) \frac{1}{v} \left(\int_0^T (T-t) \tilde{\sigma}_i^{(0)}(t)dt\right) x \\ A_3 &= (-\lambda h_i) \mathbf{E}\left[\int_0^T (h_i S_i(0)N(t))dt|\Lambda = (k, vx)\right] \\ &= (-\lambda S_i(0)h_i^2) \int_0^T \mathbf{E}[N(t)|N(T) = k]dt \\ &= (-\lambda S_i(0)h_i^2) \frac{(kT)}{2}. \end{aligned}$$

Here we have used the fact that $(N(t)|N(T) = k)$ is a binomial variable with k independent 0-1 trials and probability $\frac{\lambda t}{\lambda T} = \frac{t}{T}$ of taking 1, which implies $\mathbf{E}[N(t)|N(T) = k] = \frac{kt}{T}$. The second term can be written as

$$\mathbf{E}\left[\int_0^T \sigma_i^{(1)}(t)S_i^{(1)}(t)dW_i(t)|\Lambda = (k, vx)\right] = B_1 + B_2 + B_3,$$

where

$$\begin{aligned}
B_1 &= \mathbf{E} \left[\int_0^T \sigma_i^{(1)}(t) (-\lambda h_i S_i(0)t) dW_i(t) | \Lambda = (k, vx) \right] \\
&= (-\lambda h_i) S_i(0) \left(\int_0^T t \tilde{\sigma}_i^{(1)}(t) dt \right) \frac{1}{v} x \\
B_2 &= \mathbf{E} \left[\int_0^T \sigma_i^{(1)}(t) \left(\int_0^t \sigma_i^{(0)}(s) dW_i(s) \right) dW_i(t) | \Lambda = (k, vx) \right] \\
&= \frac{1}{v^2} \left(\int_0^T \left(\int_0^t \tilde{\sigma}_i^{(0)}(s) ds \right) \tilde{\sigma}_i^{(1)}(t) dt \right) (x^2 - 1) \\
B_3 &= \mathbf{E} \left[\int_0^T \sigma_i^{(1)}(t) (h_i S_i(0) N(t)) dW_i(t) | \Lambda = (k, vx) \right] \\
&= h_i S_i(0) \mathbf{E} \left[\int_0^T \sigma_i^{(1)}(t) \frac{kt}{T} dW_i(t) | \Delta(T) = vx \right] \\
&= h_i S_i(0) \frac{k}{T} \left(\int_0^T t \tilde{\sigma}_i^{(1)}(t) dt \right) \frac{1}{v} x.
\end{aligned}$$

The computation of B_2 is discussed in Kunitomo and Takahashi (2001, Lemma A.1). The third term can be written as

$$h_i \mathbf{E} \left[\int_0^T S_i^{(1)}(t-) dN(t) | \Lambda = (k, vx) \right] = C_1 + C_2 + C_3,$$

where

$$\begin{aligned}
C_1 &= h_i \mathbf{E} \left[\int_0^T \left(-\lambda h_i S_i(0)t \right) dN(t) \Big| \Lambda = (k, vx) \right] \\
&= -\frac{1}{2} S_i(0) \lambda h_i^2 T k \\
C_2 &= h_i \mathbf{E} \left[\int_0^T \left(\int_0^t \sigma_i^{(0)}(s) dW_i(s) \right) dN(t) \Big| \Lambda = (k, vx) \right] \\
&= h_i \mathbf{E} \left[\int_0^T \left(\int_0^t \sigma_i^{(0)}(s) dW_i(s) \right) \frac{k}{T} dt \Big| \Delta(T) = vx \right] \\
&= h_i \frac{k}{T} \left(\int_0^T (T-t) \tilde{\sigma}_i^{(0)}(t) dt \right) \frac{1}{v} x \\
C_3 &= h_i \mathbf{E} \left[\int_0^T \left(h_i S_i(0) N(t-) \right) dN(t) \Big| \Lambda = (k, vx) \right] \\
&= h_i^2 S_i(0) \mathbf{E} \left[\sum_{l=0}^{N(T)-1} l \Big| N(T) = k \right] \\
&= h_i^2 S_i(0) \frac{k^2 - k}{2}.
\end{aligned}$$

Substituting the first and second order conditional expectations into (9) we get the lower bound approximation as

$$LBA = \sum_{k=0}^{\infty} p_k \int_{-\infty}^{\infty} \left(cx^2 + a_1(k)x + a_0(k) \right)^+ d\Phi(x), \quad (10)$$

where

$$\begin{aligned}
c &= \frac{1}{v^2} \sum_{i=1}^n w_i \left(\int_0^T \left(\int_0^t \tilde{\sigma}_i^{(0)}(s) ds \right) \tilde{\sigma}_i^{(1)}(t) dt \right) \\
a_0(k) &= S(0) + (K - \lambda T) \sum_{i=1}^n w_i h_i S_i(0) + \frac{1}{2} \sum_{i=1}^n w_i S_i(0) h_i^2 ((k - \lambda T)^2 - k) - c - K \\
a_1(k) &= v + \frac{1}{v} \sum_{i=1}^n w_i h_i \left(\frac{k}{T} - \lambda \right) \left(\int_0^T ((T - t) \tilde{\sigma}_i^{(0)}(t) + S_i(0) t \tilde{\sigma}_i^{(1)}(t)) dt \right).
\end{aligned}$$

Note that in computing the lower bound approximation (10) there is no need to do numerical integration. This is because for every fixed k the integrand is the positive part of a quadratic function which contains only three cases: no root, one root, and two roots. Therefore the computation of the lower bound approximation *LBA* is easy and fast. This is one key advantage over the other methods such as simulation or finite difference for the PIDE.

4 Numerical Results

In this section we conduct some numerical tests for European basket calls with underlying asset price processes satisfying (1) to test the performance of the lower bound approximation. The Monte Carlo simulation provides the benchmark results. The control variate technique is adopted to reduce the standard deviations. The following data are used in all numerical tests: the number of assets in the basket $n = 4$, the portfolio weights of each asset $w_i = 0.25$ for $i = 1, \dots, n$, the correlation coefficients of Brownian motions $\rho_{ij} = 0.3$ for $i, j = 1, \dots, n$, the initial asset prices $S_i(0) = 100$ for $i = 1, \dots, n$, the exercise price $K = 100$, and the jump intensity $\lambda = 0.3$.

Tables 1 displays the European basket option values with four different methods: the Monte Carlo (MC), the partial exact approximation (PEA), the asymptotic expansion (AE) and the lower bound approximation (LBA). The local volatility function is $\sigma(t, S) = 0.2S$. We perform numerical tests for three constant jump sizes $h_i = h = -0.2212, -0.1175, -0.0606$. It is clear that all three approximation methods perform well with relative errors less than 1%. We have done other tests with the same data except the volatility function being changed to $\sigma(t, S) = 0.5S$. The overall relative error is 0.6% for the PEA, 4% for the AE and 1.7% for the LBA. Tables 1 shows that the performance of the PEA is the best while the LBA performs better than the AE. The results suggest that the PEA is the best approximation method for the European basket call pricing when local volatility functions are of the Black-Scholes type. On the other hand, the AE and LBA are much more flexible and can handle general local volatility functions. The LBA has the additional advantage of having closed-form solutions when the local volatility functions are time independent and can deal with different jump sizes to common jumps. The AE is slow in solving the PIDE with the finite difference method and requires the same jump size of all assets to common jumps.

Table 2 displays the numerical results with the MC, AE and LBA and with different maturities ($T = 1, 3$) and local volatility functions ($\sigma(S) = \alpha S^\beta$ with $\alpha = 0.2, 0.5$ and $\beta = 1, 0.8, 0.5$). The jump size is $h_i = h = -0.2212$ for all assets. The last row displays the average errors of the AE and LBA. Table 2 shows that the performance of the LBA is excellent with the average relative error 0.4%. The overall performance of the LBA is better than that of the AE, especially when the local volatility function is $\sigma(S) = 0.5S$. Matlab is used for computation. The LBA only takes a few seconds for each case, much faster than the AE and MC. We have done other tests with the same data as in Table 2 except different intensity rates ($\lambda = 0$ and 1). The overall performance is essentially the same as that of Table 2. We can say with reasonable confidence that the LBA suggested in this paper works well.

Tables 3 displays the numerical results with the MC and LBA and with different maturities T and local volatility functions $\sigma(t, S) = \alpha S^\beta$. The jump sizes for four assets are $h_1 = 0$, $h_2 = 0.3$, $h_3 = -0.3$, and $h_4 = 0$, which implies that the jump event has no impact to asset prices 1 and 4, positive impact to asset price 2 and negative impact to asset price 3. Since The PEA requires the Black-Scholes setting while the AE requires the same jump size for all assets, neither method can be used in this test. It is clear that the LBA performs well in comparison with the MC with the overall relative error less than 1%.

5 Conclusion

In this paper we discuss Rogers and Shi's lower bound approximation (LBA) to basket option values for local volatility jump-diffusion models. We expand parameterized asset prices to the second order with the asymptotic expansion method and obtain an easily implemented LBA. It turns out that if the local volatility function is time independent, such as the CEV model, then there is a closed-form expression for the approximation. We compare the numerical performance of the LBA with other methods using different parameters and show that the LBA is fast and accurate in most cases.

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λ	h	T	MC (stdev)	PEA (err%)	AE (err%)	LBA (err%)
0.3	-0.2212	1	7.35 (0.01)	7.35 (0.0)	7.35 (0.0)	7.37 (0.3)
		3	12.93 (0.01)	12.92 (0.1)	12.85 (0.6)	12.86 (0.5)
	-0.1175	1	6.08 (0.01)	6.08 (0.0)	6.07 (0.2)	6.09 (0.2)
		3	10.57 (0.01)	10.56 (0.1)	10.49 (0.8)	10.57 (0.0)
	-0.0606	1	5.66 (0.01)	5.66 (0.0)	5.65 (0.2)	5.67 (0.2)
		3	9.83 (0.01)	9.82 (0.1)	9.74 (0.9)	9.86 (0.3)
1	-0.2212	1	10.78 (0.01)	10.77 (0.1)	10.78 (0.0)	10.82 (0.4)
		3	18.64 (0.01)	18.63 (0.1)	18.57 (0.4)	18.91 (1.4)
	-0.1175	1	7.28 (0.01)	7.28 (0.0)	7.28 (0.0)	7.31 (0.4)
		3	12.65 (0.01)	12.64 (0.1)	12.58 (0.6)	12.68 (0.2)
	-0.0606	1	6.02 (0.01)	6.02 (0.0)	6.01 (0.2)	6.03 (0.2)
		3	10.45 (0.01)	10.43 (0.2)	10.37 (0.8)	10.47 (0.2)
Average			(0.1)	(0.4)	(0.4)	

Table 1: The comparison of European basket call option prices with the Monte Carlo (MC), the partial exact approximation (PEA), the asymptotic expansion (AE) and the lower bound approximation (LBA). The table displays the results with different jump intensities λ , jump sizes $h_i = h$, maturities T , and local volatility function $\sigma_i(t, S) = 0.2S$. The numbers inside brackets in the MC columns are the standard deviations and those in the PEA, AE and LBA columns are the relative percentage errors in comparison with the MC results.

T	α	β	MC (stdev)	AE (err%)	LBA (err%)
1	0.2	1	7.35 (0.01)	7.35 (0.0)	7.37 (0.3)
	0.5		14.71 (0.01)	14.42 (2.0)	14.87 (1.1)
	0.2	0,8	5.31 (0.01)	5.33 (0.4)	5.31 (0.0)
	0.5		7.33 (0.01)	7.33 (0.0)	7.34 (0.1)
	0.2	0.5	5.09 (0.01)	5.09 (0.0)	5.08 (0.2)
	0.5		5.11 (0.01)	5.12 (0.2)	5.11 (0.0)
3	0.2	1	12.93 (0.01)	12.85 (0.6)	12.86 (0.5)
	0.5		25.69 (0.04)	24.14 (6.0)	26.16 (1.8)
	0.2	0.8	9.61 (0.01)	9.64 (0.3)	9.63 (0.2)
	0.5		12.86 (0.01)	12.86 (0.0)	12.81 (0.4)
	0.2	0.5	8.96 (0.01)	8.98 (0.2)	8.91 (0.6)
	0.5		9.18 (0.01)	9.21 (0.3)	9.18 (0.0)
Average			(0.7)	(0.4)	

Table 2: The comparison of European basket call option prices with the MC, AE and LBA. The table displays results with different maturities T , local volatility functions $\sigma_i(t, S) = \alpha S^\beta$ and jump sizes $h_i = h = -0.2212$.

T	α	β	MC (stdev)	LBA (err%)	
1	0.2	1	5.53 (0.01)	5.52 (0.2)	
	0.5		13.87 (0.01)	13.95 (0.6)	
	0.2	0.8	2.22 (0.01)	2.22 (0.0)	
	0.5		5.50 (0.01)	5.49 (0.2)	
	0.2	0.5	0.63 (0.01)	0.63 (0.0)	
	0.5		1.42 (0.01)	1.42 (0.0)	
	3	0.2	1	9.68 (0.02)	9.66 (0.2)
		0.5		24.42 (0.06)	24.84 (1.7)
0.2		0.8	3.95 (0.01)	3.94 (0.3)	
0.5			9.57 (0.02)	9.59 (0.2)	
0.2		0.5	1.37 (0.01)	1.37 (0.0)	
0.5			2.59 (0.01)	2.59 (0.0)	
Average			(0.3)		

Table 3: The comparison of European basket call option prices with the MC and LBA. The table displays results with different maturities T , local volatility functions $\sigma_i(t, S) = \alpha S^\beta$, and jump sizes $h_1 = 0$, $h_2 = 0.3$, $h_3 = -0.3$, and $h_4 = 0$.